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Symmetry breaking in an anisotropic space–time

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Abstract. We calculate the one-loop effective Lagrangian for a self-interacting $\lambda\phi^4$ scalar field in a static Mixmaster universe. We analyse the effect of anisotropy on the process of symmetry restoration.

1. Introduction

One of the more popular areas of research in the last year or so has been in identifying how space–time curvature or topology can affect symmetry restoration or topological mass generation (Gibbons 1978, Shore 1979, Toms 1980, Kennedy 1981, Denardo and Spallucci 1981, Critchley *et al* 1981). So far all results have either been for flat space with non-trivial topology or for isotropic space–times with constant curvature (de Sitter or Einstein). It is the purpose of this paper to show how one can obtain useful results for the special case of an Einstein universe with a 'squashed' spatial section. This universe is interesting partly because of the anisotropy but also because it is not conformally flat.

2. Calculation

In a previous paper (Critchley and Dowker 1981) we considered free scalar fields (massive and massless) in the static Mixmaster universe in some detail and we shall therefore be somewhat brief here. The main point we shall make is that if we start from the Lagrangian

$$\mathscr{L} = -\tfrac{1}{2}\tilde{\phi}(x)(\Box + \xi_{\mathrm{B}}R + m_{\mathrm{B}}^2)\tilde{\phi} - \tfrac{1}{24}\lambda_{\mathrm{B}}\tilde{\phi}^4 \tag{1}$$

and write $\tilde{\phi} = \hat{\phi} + \phi$ where ϕ is a small quantum perturbation about the classical condensate $\hat{\phi}$, then the propagator for ϕ, $G(x, x')$ satisfies

$$(\Box + m_{\mathrm{B}}^2 + \xi_{\mathrm{B}}R + \tfrac{1}{2}\lambda_{\mathrm{B}}\hat{\phi}^2)G(x, x') = \delta(x, x') \tag{2}$$

since $\hat{\phi}$ is a constant for static spatially homogeneous space–times. This is merely the Klein–Gordon equation for a scalar field of mass $\mu_{\mathrm{B}}^2 = m_{\mathrm{B}}^2 + \tfrac{1}{2}\lambda_{\mathrm{B}}\hat{\phi}^2$ and we can use all our free-field results.

For simplicity we shall assume m_{B}^2, $(\xi_{\mathrm{B}} - \tfrac{1}{6})$ and λ_{B} are small, and therefore that we can expand the effective Lagrangian in a Taylor series about small $\mu_{\mathrm{B}}^2 + (\xi_{\mathrm{B}} - \tfrac{1}{6})R$. Our version of the zeta function regularisation scheme involves finding the solution to

$$(\Box + \mu_{\mathrm{B}}^2 + \xi_{\mathrm{B}}R)\zeta_4(\nu) = \zeta_4(\nu - 1). \tag{3}$$

Here $\zeta_4(\nu)$ is related to the zeta function on the spatial section by (Dowker and Kennedy 1978)

$$\zeta_4(\nu)=\frac{\mathrm{i}}{\sqrt{4\pi}}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\zeta_3(\nu-\tfrac{1}{2}). \tag{4}$$

$\zeta_4(\nu)$ can easily be related to the Green function $G(x, x')$ (equation (2)) if we note that (3) has the solution

$$\zeta_4(\nu)=\frac{\mathrm{i}^\nu}{\Gamma(\nu)}\int_0^\infty \frac{\mathrm{d}s}{s^{1-\nu}}\exp(-\mathrm{i}\mu^2 s)\langle x'(s)|x''(0)\rangle_4 \tag{5}$$

where $\langle x'(s)|x''(0)\rangle$ is independent of ν. Clearly from (5) we can write

$$\zeta_4(\nu)=\frac{(-1)^{\nu-1}}{\Gamma(\nu)}\left(\frac{\mathrm{d}}{\mathrm{d}\mu^2}\right)^{\nu-1}G(x,x'). \tag{6}$$

The metric for the static Mixmaster universe is given by (Hu 1973)

$$(\mathrm{d}s)^2=(\mathrm{d}t)^2-1_1^2(\mathrm{d}\theta)^2-1_3^2(\mathrm{d}\psi)^2-21_3^2\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\psi-(\sin^2\theta 1_1^2+1_3^2\cos^2\theta)(\mathrm{d}\phi)^2. \tag{7}$$

Note that we have retained a certain amount of symmetry. Equation (2) has the solution

$$G_4=\frac{\mathrm{i}}{2V}\sum_{J=0}^{\infty}(2J+1)\sum_{m=-J}^{J}\sum_{n=-J}^{J}\frac{\exp[-\mathrm{i}E_{l,m}(t-t')]}{E_{l,m}}D^l_{mn}(\theta)\bar{D}^l_{mn}(\theta')$$
$$\times\exp[\mathrm{i}m(\psi-\psi')]\exp[\mathrm{i}n(\phi-\phi')] \tag{8}$$

with

$$E^2_{l,m}=\frac{1}{1_1^2}\left[\frac{l^2}{4}+\alpha\left(m^2+\frac{1}{12(1+\alpha)}\right)\right]+\mu_{\mathrm{B}}^2$$
$$V=16\pi^2 1_1^2 1_3\simeq 2\pi^2 a^3(1-\tfrac{1}{2}\alpha+\tfrac{3}{8}\alpha^2) \tag{9}$$

and

$$1+\alpha=1_1^2/1_3^2.$$

The sum over J covers half-integral and integral values which correspond to a sum over l ($\equiv 2J+1$) in equation (8). The derivation of mode functions, energies and further details about the space–time can be found in Hu (1973).

The zeta function equation (6) reduces to

$$\zeta_4(\nu)=\frac{\mathrm{i}2^{1-\nu}(-1)^{\nu-1}}{2V\Gamma(\nu)}\sum_{J=0}^{\infty}(2J+1)\sum_{-J}^{J}\sum_{-J}^{J}\frac{D^l_{mn}(\theta)\bar{D}^l_{mn}(\theta')}{(E_{l,m})^{2\nu-1}}\exp[\mathrm{i}m(\psi-\psi')]\exp[\mathrm{i}n(\phi-\phi')]$$
$$\times\sum_{n=0}^{\infty}(-\mathrm{i})^n\frac{(t-t')^n}{n!}E^n_{J,K}(n-1)(n-3)\ldots(n+3-2\nu). \tag{10}$$

The properties of the rotation matrices $D^l_{mn}(\theta)$ can be found in Vilenkin (1968). Although (10) has been derived for integral ν we can easily rewrite it in terms of gamma functions and thereby allow the ν to become arbitrary. From (5) we can derive the relation

$$\frac{\mathrm{d}}{\mathrm{d}\mu^2}\zeta_4(\nu)=\frac{-\Gamma(\nu+1)}{\Gamma(\nu)}\zeta_4(\nu+1) \tag{11}$$

which is exceedingly useful in deriving our Taylor series. Accordingly, we only need to know the massless value of $\zeta_4(\nu)$ or alternatively $\zeta_3(\nu)$. From Critchley and Dowker (1981), or of course via equations (4) and (10), we find

$$\zeta_3(\nu)=a^{2\nu}\{\zeta_R(2\nu-2)-\tfrac{1}{3}\alpha\nu\zeta_R(2\nu-2)+4\alpha^2[\tfrac{1}{12}\nu\zeta_R(2\nu)] + 2\nu(\nu+1)[\tfrac{1}{80}\zeta_R(2\nu-2)-\tfrac{1}{36}\zeta_R(2\nu)+\tfrac{1}{45}\zeta_R(2\nu+2)]\}+O(\alpha^3). \tag{12}$$

$\zeta_R(\nu)$ is the usual Riemann zeta function. In principle there is nothing to stop us going to arbitrary order in the distortion parameter α, but we shall stop at quadratic order.

The effective Lagrangian is given by

$$\mathscr{L}=\lim_{\nu\to 1}\frac{-\mathrm{i}}{2(\nu-1)}\zeta_4(\nu-1)L^{2-2\nu} \tag{13}$$

and can be expanded in a Taylor series to give (using equations (10)–(13))

$$\begin{aligned}\mathscr{L}=&\frac{\alpha^2}{180\pi^2a^4}\left[\Omega+\ln\left(\frac{a^2}{L^2}\right)\right]-\frac{1}{480\pi^2a^4}(1+\tfrac{2}{3}\alpha+\tfrac{47}{45}\alpha^2)\\ &+\frac{[m_B^2+\frac{1}{2}\lambda_B\hat{\phi}^2+(\xi_B-\frac{1}{6})R]}{96\pi^2a^2}[1+\tfrac{1}{6}\alpha+\tfrac{8}{5}\alpha^2(\tfrac{1}{3}-\zeta_R(3))]\\ &+\frac{[(\xi_B-\frac{1}{6})R+m_B^2+\frac{1}{2}\lambda_B\hat{\phi}^2]^2}{64\pi^2}\left[\Omega+\ln\left(\frac{a^2}{L^2}\right)(\tfrac{1}{2}\alpha-\tfrac{7}{30}\alpha^2)\right.\\ &\left.+\tfrac{2}{3}\alpha^2(-\zeta_R(3)+2\zeta_R(5))\right]+\dots\end{aligned} \tag{14}$$

$$\Omega=\frac{1}{s-1}+2-2\ln 2+2\gamma.$$

Since this is ostensibly a massive theory we are permitted to absorb $\ln|m^2L^2|$ terms into a renormalisation of the coupling constants, e.g. Bunch and Davies (1978).

So if we define $\bar{\Omega}=1/(s-1)+2-2\ln 2+2\gamma-\ln|m^2L^2|$ then the $\bar{\Omega}$ terms can be interpreted as a renormalisation of the classical Lagrangian which is given by

$$\mathscr{L}_{\mathrm{cl}}=-\Lambda_B+\frac{R}{16\pi G_B}+\bar{\alpha}_B^2(R^2-3R_{\mu\nu}R^{\mu\nu})-\tfrac{1}{2}(\hat{\phi}\Box\hat{\phi}+m_B^2\hat{\phi}^2+\xi_BR\hat{\phi}^2+\tfrac{1}{12}\lambda_B\hat{\phi}^4). \tag{15}$$

In the usual way we find

$$\Lambda_B=\Lambda+\frac{m^4}{32\pi^22}\bar{\Omega}$$

$$\frac{R}{16\pi G_B}=\frac{R^2}{16\pi G}-\frac{m^2}{16\pi^22}(\xi-\tfrac{1}{6})R\bar{\Omega}$$

$$\bar{\alpha}(R^2-3R_{\mu\nu}R^{\mu\nu})=\bar{\alpha}_B(R^2-3R_{\mu\nu}R^{\mu\nu})+\frac{\alpha^2\bar{\Omega}}{180\pi^2a^4}+\frac{(\xi_B-\frac{1}{6})^2}{64\pi^2a^4}36(1+\tfrac{2}{3}\alpha-\tfrac{5}{9}\alpha^2)\bar{\Omega} \tag{16}$$

and

$$m_B^2=m^2+\frac{m^2\lambda}{32\pi^2}\bar{\Omega}\qquad \lambda_B=\lambda+\frac{3\lambda^2}{32\pi^2}\bar{\Omega}\qquad \xi_B=\xi+\frac{\lambda}{32\pi^2}(\xi-\tfrac{1}{6})\bar{\Omega} \tag{17}$$

which leaves

$$\mathscr{L}_{\rm ren}=\frac{\alpha^2}{180\pi^2a^4}\ln|a^2m^2|-\frac{1}{480\pi^2a^4}(1+\tfrac{2}{3}\alpha+\tfrac{47}{45}\alpha^2)+\frac{[m^2+\frac12\lambda\hat\phi^2+(\xi-\frac16)R]}{96\pi^2a^2}$$
$$\times[1+\tfrac16\alpha+\tfrac85\alpha^2(\tfrac13-\zeta_{\rm R}(3))]+\frac{[m^2+(\xi-\frac16)R+\frac12\lambda\hat\phi^2]^2}{.64\pi^2}$$
$$\times[\ln|a^2m^2|+(-\tfrac13\alpha+\tfrac{7}{30}\alpha^2)+\tfrac23\alpha^2(-\zeta_{\rm R}(3)+2\zeta_{\rm R}(5))]. \tag{18}$$

Defining $\mathscr{L}_{\rm tot}=\mathscr{L}_{\rm cl}+\mathscr{L}_{\rm ren}$ where $\mathscr{L}_{\rm cl}$ is given by (15), but with the subscript B dropped, we now look for a solution to the equation

$$\frac{\partial}{\partial\hat\phi}\mathscr{L}_{\rm tot}|_{\hat\phi\neq0}=0 \tag{19}$$

for some (real) value of $\hat\phi$.

It is easy to show that a solution is possible provided

$$m^2+\xi R<\frac{\lambda\hbar}{96\pi^2a^2}[1+\tfrac16\alpha+\tfrac85\alpha^2(\tfrac13-\zeta_{\rm R}(3))]+\frac{\lambda\hbar}{32\pi^2}[m^2+(\xi-\tfrac16)R]$$
$$\times[\ln|m^2a^2|-\tfrac13\alpha+\tfrac{7}{30}\alpha^2+\tfrac23\alpha^2(\zeta_{\rm R}(3)+2\zeta_{\rm R}(5))]+{\rm O}(\hbar^2) \tag{20}$$

where terms of order $[m^2+(\xi-\frac16)R]^2$ have been dropped.

In order to determine the critical value of m^2, denoted by $m_{\rm C}^2$, for which the inequality holds, we approximate the value of $m_{\rm C}^2$ on the right-hand side of equation (20) by its classical value $m_{\rm C}^2=-\xi R$. This involves no loss of generality since all we are doing is throwing away terms of order $(\hbar^2)$. The resultant $\ln\xi R$ term can then be expanded in terms of $\xi-\frac16$ and the α up to the required order. Our final result is

$$m_{\rm C}^2+\xi R=\frac{\lambda\hbar}{96\pi^2a^2}[1+\tfrac16\alpha+\tfrac15\alpha^2(5+2\zeta_{\rm R}(3)-20\zeta_{\rm R}(5))]$$
$$-\frac{\lambda\hbar}{32\pi^2}(\xi-\tfrac16)R+{\rm O}(\hbar^2)+{\rm O}(\alpha^3)+{\rm O}[(\xi-\tfrac16)^2]. \tag{21}$$

It is of no little interest to note that equations (18) and (20) are real even when $m^2<0$ and $\hat\phi=0$. This is not the case for an analogous situation (flat space, finite temperature) discussed by Dolan and Jackiw (1974). These authors computed the critical temperature at which the symmetry is spontaneously broken and found contributions which were imaginary and could not be included in the calculation of the critical temperature. For comparison, we quote their result for their effective potential which is the negative of the effective Lagrangian. They find,

$$V(\phi^2)=-\frac{1}{64\pi^2}M^4\ln\beta^2|m^2|-\tfrac32(M^2-\tfrac23m^2)^2-\frac{\pi^2}{\beta^4}+\frac{M^2}{24\beta^2}$$
$$-\frac{1}{12\pi}\frac{M^3}{\beta}+\frac{1}{64\pi^2}(\tfrac32+2\ln4\pi-2\gamma)M^4+{\rm O}(m^6\beta^2) \tag{22}$$

where $M^2=(m^2+\frac12\lambda\phi^2)$, $\beta=(\text{temperature})^{-1}$.

Equation (22) is a high-temperature expansion. Note the error in their equation (3.15): $\ln M^2/m^2$ should be $\ln M^2/|m^2|$, the $|m^2|$ coming from a (real) renormalisation of the cosmological constant.

3. Comments

If grand unified theories have a role to play in the early history of the universe, then it will be necessary to give a systematic treatment of symmetry breaking/restoration in curved spaces (and at finite temperatures). The present calculation, although somewhat trivial, gives an indication of a trend that can be expected from a spatial anisotropy. It should be possible to allow for this, at least in those scenarios that have an Einstein phase.

In particular, we observe from (21) that the classical and quantum corrections are in competition. For the prolate situation $\alpha > 0$, the effect of the classical ξR term is to enhance symmetry restoration, whereas the quantum corrections will try to suppress restoration—at least to first order in α and with conformal coupling.

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